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Dirac Operators and Clifford Geometry - New Unifying Principles in Particle Physics ?

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Abstract. In this lecture I will report on some recent progress in understanding the relation of Dirac operators on Clifford modules over an even-dimensional closed Riemannian manifold M and (euclidean) Einstein-Yang-Mills-Higgs models.

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Although being a gauge theory, it is well-known that the classical theory of gravity as enunciated by Einstein stands apart from the non-abelian gauge field theory of Yang, Mills and Higgs, which encompasses the three other fundamental forces: the electromagnetic, weak and the strong interaction. General relativity is governed by a variational principle associated with the Lagrangian

$$S_{GR} = \frac{1}{16\pi \mathrm{G}} \int_M *r_M \tag{1}$$

where G denotes Newton's constant, $*$ is the Hodge star and r_M is the scalar curvature of the space-time M, a closed four-dimensional pseudo-Riemannian manifold of signature $(-, + \cdots +)$. To describe a Yang-Mills-Higgs model the space-time manifold M, apart from the metric, is endowed with additional structure (cf. [IS]): Over M we have both a principal bundle P_G with structure group a compact Liegroup G and a Clifford module $\mathcal{E}$ that is assumed to furnish a representation $\rho: \mathcal{G} \to \mathrm{Aut}_{C(M)}\mathcal{E}$ of the group of gauge transformations $\mathcal{G}$ of P_G. Here $Aut_{C(M)}\mathcal{E}$ are those automorphisms of $\mathcal{E}$ which commute with the Clifford action. Let A be a connection on P_G with curvature $R \in \Omega^2(M,\, ad(P_G))$ [1]), $\nabla^{\mathcal{E}}: \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$ be the associated Clifford connection and $\varphi \in \Gamma(W)$ a section of an additional vector bundle W associated to P_G with induced connection ∇^W. Then a corresponding Yang-Mills-Higgs model is based on the Lagrangian

$$S_{YMH} = -\frac{1}{g^2}\int_M tr\big(R \wedge *R\big) + \int_M \Big(\big(\nabla^W \varphi \wedge *\nabla^W \varphi\big) + *V(\varphi)\Big) + \int_M *(\psi, D_\varphi \psi)_{\mathcal{E}} \tag{2}$$

where $V: W \to \mathbb{C}$ is an invariant quartic polynomial, $\psi \in \Gamma(\mathcal{E})$ and $D_\varphi: \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ de-

[1]) Here $ad(P_G)$ denotes the vector bundle with fibre the Lie algebra LG associated to P_G with respect to the adjoint representation $Ad: G \to LG$.

notes a Dirac-Yukawa operator associated to $(\nabla^{\mathcal{E}},\ \varphi)$. The constant g in (2) parametrizes the fibre metric $tr\colon ad(P_G)\times ad(P_G)\to\mathbb{C}$ which is induced by the Killing form on the Lie algebra LG of G and is called the Yang-Mills coupling constant. If G is not simple, it is possible to generalize (2) introducing a seperate coupling constant for each simple factor.

With exception of the 'pure' Yang-Mills term $S_{YM}=-\frac{1}{g^2}\int_M R\wedge *R$, from a mathematical point of view the lagrangian (2) looks highly artificial. So we briefly comment on its physical significance:

- The bosonic part of a Yang-Mills-Higgs model which describes (nonabelian) gauge forces, is defined by the first two terms of (2). The (covariant) Klein-Gordon lagrangian $S_{KG}=\int_M \nabla^W\varphi\wedge *\nabla^W\varphi$ and the Higgs potential $S_\varphi=\int_M *V(\varphi)$ for the Higgs field $\varphi\in\Gamma(W)$ are added to the pure Yang-Mills part such that the gauge bosons or connections $A^i\in\Omega^1(M,ad(P_G))$ acquire masses. In the theory of electroweak interaction for example, where we have $G_{\rm ew}=SU(2)\times U(1)$ and $V(\varphi):=\frac{\lambda}{4}(\varphi,\varphi)^2-\frac{\mu^2}{2}(\varphi,\varphi)$ with $\varphi\in\Gamma(M\times\mathbb{C}^2)$, $(\,\cdot\,,\,\cdot\,)$ is the standard inner product on $\mathbb{C}^2$ and $\lambda,\mu>0$.
- The fermionic part $\int_M *(\psi,D_\varphi\psi)_{\mathcal{E}}$ which describes matter fields[2]) is defined by the Dirac-Yukawa operator $D_\varphi:=D+c_Y\widetilde{\phi}(\varphi)$. Here D is the Dirac operator corresponding to the Clifford connection $\nabla^{\mathcal{E}}$ and $\widetilde{\phi}\colon W\to{\rm End}^-_{C(M)}\mathcal{E}$ is a linear map. The Yukawa coupling $c_Y\in\mathbb{R}$ gives rise to the fermion mass as soon as there exists a non-vanishing $\varphi_0\in\Gamma(W)$ which minimizes the Higgs-potential V but is only invariant under a subgroup $\mathcal{H}\subset\mathcal{G}$ of the group of gauge transformations corresponding to a subgroup H of the gauge group G. For simplicity here we have assumed only one fermion generation [3]).

The mass acquisition for the gauge bosons as well as for the fermions of the theory by introducing the Higgs field $\varphi\in\Gamma(W)$ is called 'spontaneous symmetry breaking'. For a mathematical audience it might be considered as a physical counterpart of reducing the G-principal bundle P_G, c.f. [NS]. It plays a central rôle in the Weinberg-Salam model of electroweak interaction where it produces three massive and one massless boson - the $W^+,\ W^-,\ Z$ and the photon - and gives also masses to the electron, the muon and the tauon but not to the corresponding neutrinos.

As a particular Yang-Mills-Higgs model based on the gauge group $G=SU(3)\times SU(2)\times U(1)$, nowadays the Standard Model of elementary particles is extraordinarily successful in describing particle phenomenology. Nonetheless it suffers both from a technical and concepual side: There are so many variables (corresponding to coefficients in (2)) - eighteen in sum, among them for example the gauge couplings, the masses of the bosons and fermions etc. - which have to be experimentally determined and put into the model. Conceptually, to mention only one problem, it is not clear how to treat gravity in this con-

[2]) For example, we have leptons and quarks in the Standard Model.

[3]) In general, with $N\in\mathbb{N}$ fermion generations, the Clifford module $\mathcal{E}$ corresponds to $\mathcal{E}\otimes\mathbb{C}^N$. Thus, $c_Y\in{\rm End}(\mathbb{C}^N)$ mixing the different fermion generations and we obtain $\phi=\widetilde{\phi}(\varphi)\otimes c_Y$ for the Yukawa coupling. In the Weinberg-Salam model of electroweak interaction we have $N=3$ corresponding to the three lepton families of the electron e, muon μ and the tauon τ.

text. In principle one should study an Einstein-Yang-Mills-Higgs model based on the combined lagrangian

$$S = S_{GR} \; + \; S_{YMH}. \tag{3}$$

The corresponding Euler-Lagrange variational equations are coupled equations for the gravitational and all bosonic and fermionic fields. Because nowadays the best colliders in high-energy physics are only able to observe scattering processes on a scale determined by the masses of the heaviest fermions, for example by the top-quark, and Newton's constant G is so tiny compared to this scale, unfortunately gravitational effects among individual particles are undetectable. However there are other experimental data which seem to justify this 'Ansatz', cf. [D].

In this lecture I will report on some recent progress in understanding the relationship of Dirac operators and *euclidean* Einstein-Yang-Mills-Higgs models. So we deal with a closed Riemannian manifold M of even dimension n and a Clifford module $\mathcal{E}$ over M furnished with a Clifford connection $\nabla^{\mathcal{E}}\colon \Gamma(\mathcal{E}) \to \Gamma(T^*M \otimes \mathcal{E})$ with twisting curvature $R^{\mathcal{E}/S} \in \Omega^2(M, \mathrm{End}_{C(M)}\mathcal{E})$. Tensoring $\mathcal{E}$ with the $\mathbb{Z}_2$-graded trivial vector bundle $\mathbb{C}^{1|1}$ whose even and odd subbundles each have complex rank one, obviously we obtain the Clifford module $\bar{\mathcal{E}} := \mathcal{E} \otimes \mathbb{C}^{1|1}$. Furthermore denote by $\Psi\mathrm{DO}(\bar{\mathcal{E}})$ the space of pseudo-differential operators on $\bar{\mathcal{E}}$. Then, one observes the

Theorem 1. *There exists a Dirac operator $D_\phi\colon \Gamma(\bar{\mathcal{E}}) \to \Gamma(\bar{\mathcal{E}})$ on $\bar{\mathcal{E}} := \mathcal{E} \otimes \mathbb{C}^{1|1}$ such that*

$$res(D_\phi^{-n+2}) \sim a_0 \int_M *r_M - a_1 \int_M tr\big(R^{\mathcal{E}/S} \wedge * R^{\mathcal{E}/S}\big) + a_1 \int_M tr\big(\nabla^{\mathcal{E}}\phi \wedge * \nabla^{\mathcal{E}}\phi\big) + \frac{a_1}{2} \int_M * V(\phi)$$

where $res\colon \Psi\mathrm{DO}(\bar{\mathcal{E}}) \to \mathbb{C}$ denotes the non-commutative residue, $\phi \in \mathrm{End}^-_{C(M)}\mathcal{E}$, $V(\phi) = tr(\phi^4 - \phi^2)$ and the coefficients are $a_o = \frac{1}{12}$ and $a_1 = \frac{2}{\mathrm{rk}(\mathcal{E}/S)}$, respectively, where $\mathrm{rk}(\mathcal{E}/S)$ is the rank of the (virtual) twisting part $\mathcal{E}/S$ of the Clifford module $\mathcal{E}$.

Amazingly, this theorem not only offers an explanation of the origin of both gravitational and Yang-Mills gauge symmetries, but much more. Before arguing, I briefly scetch its proof (cf. [A3]):

Recall that the non-commutative residue of an operator $P \in \Psi\mathrm{DO}(E)$ on a complex vector bundle E over M can be defined by $res(P) := (2\pi)^{-n} \int_{S^*M} tr\big(\sigma^P_{-n}(x,\xi)\big) dx d\xi$, where $S^*M \subset T^*M$ denotes the co-sphere bundle on M and σ^P_{-n} is the component of order $-n$ of the complete symbol $\sigma^P := \sum_i \sigma^P_i$ of P, and is the only non-trivial trace on the algebra of pseudo-differential operators $\Psi\mathrm{DO}(E)$, cf. [W]. Given an elliptic operator P of order d and $k \in \mathbb{N}$ with $n - k > 0$, $res(P^{-\frac{(n-k)}{d}})$ can be related to the coefficient $h_k(P)$ of $t^{\frac{k-n}{d}}$ in the asymptotic expansion of $Tr\; e^{-tP}$. In particular, for a generalized laplacian $\triangle$ and $k = 2$ one obtains (cf. [KW], [A1])

$$res(\triangle^{-\frac{n}{2}+1}) \;=\; c_n \int_M *tr\big(\tfrac{1}{6} r_M \mathbb{1}_E - F\big) \tag{4}$$

with $c_n := \frac{(n-2)}{(4\pi)^{\frac{n}{2}} \cdot \Gamma(\frac{n}{2})}$, since $h_2(\triangle) = \int_M \; *tr\big(\frac{1}{6} r_M \mathbb{1}_E - F\big)$ where $F \in \mathrm{End}\; E$ orginates

from the decomposition $\triangle = \triangle^{\hat{\nabla}^E} + F$, cf. [BGV]. Here $\triangle^{\hat{\nabla}^E}$ denotes the connection laplacian corresponding to $\hat{\nabla}^E\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$.

Now let $E = \mathcal{E}$ be a Clifford module and D a Dirac operator, i.e. an odd-parity first order differential operator $D\colon \Gamma(\mathcal{E}^{\pm}) \to \Gamma(\mathcal{E}^{\mp})$ such that its square D^2 is a generalized laplacian. Using Quillen's theory of superconnections [Q] on $\mathbb{Z}_2$-graded vectorbundles [4]) it is well-known that any Clifford superconnection $\mathbb{A}$ on $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ uniquely determines a Dirac operator $D_{\mathbb{A}}$ due to the following construction

$$D_{\mathbb{A}}\colon \Gamma(\mathcal{E}) \xrightarrow{\mathbb{A}} \Omega^*(M, \mathcal{E}) \xrightarrow{\cong} \Gamma(C(M) \otimes \mathcal{E}) \xrightarrow{c} \Gamma(\mathcal{E}), \tag{5}$$

i.e. there is a one-to-one correspondence between Clifford superconnections and Dirac operators. The isomorphism is induced by the quantisation map $\mathbf{c}\colon \Lambda^*(T^*M) \xrightarrow{\cong} C(M)$ and c denotes the given Clifford action of the Clifford bundle $C(M)$ on $\mathcal{E}$. Emphasizing this approach to Dirac operators one shows the following generalization of Lichnerowicz's result (cf. [A2]):

THEOREM 2. *Let $\mathbb{A} = \mathbb{A}_{[1]} + \bar{\mathbb{A}}$ be a Clifford superconnection on a Clifford module $\mathcal{E}$ over an even-dimensional Riemannian manifold M and let $D_{\mathbb{A}} := \mathbf{c} \circ \mathbb{A}$ denote the corresponding Dirac operator. Then*

$$D_{\mathbb{A}}^2 \;=\; \triangle^{\hat{\nabla}^{\mathcal{E}}} \;+\; \frac{r_M}{4} \;+\; \mathbf{c}\big(\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}\big) \;+\; P(\bar{\mathbb{A}}) \tag{6}$$

where $\hat{\nabla}^{\mathcal{E}} := \mathbb{A}_{[1]} + \beta(\bar{\mathbb{A}})$ determines the connection laplacian $\triangle^{\hat{\nabla}^{\mathcal{E}}}$, $\mathbb{F}(\mathbb{A})^{\mathcal{E}/S}$ denotes the twisting supercurvature of $\mathbb{A}$, $P(\bar{\mathbb{A}}) := \mathbf{c}\big(\bar{\mathbb{A}}\big)^2 - \mathbf{c}\big(\bar{\mathbb{A}}^2\big) + ev_g\big(\beta(\bar{\mathbb{A}}) \cdot \beta(\bar{\mathbb{A}})\big)$ and $\beta(\bar{\mathbb{A}}) \in \Omega^1(M, \mathrm{End}\,\mathcal{E})$ is locally defined by $\beta(\bar{\mathbb{A}}) := dx^k \otimes \mathbf{c}\big(i(\partial_k)\bar{\mathbb{A}}\big)$.

Here the dot '$\cdot$' indicates the fibrewise defined product in the algebra bundle $T(M) \otimes \mathrm{End}\,\mathcal{E}$ with $T(M)$ being the tensor bundle of T^*M. The endomorphism $P(\bar{\mathbb{A}}) \in \Gamma(\mathrm{End}\mathcal{E})$ depends only on the higher degree parts $\mathbb{A}_{[i]},\ i \geq 2$ of the Clifford superconnection. For example, calculation of $P(\bar{\mathbb{A}})$ for $\mathbb{A} := \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \mathbb{A}_{[2]}$ where the two-form part is given by $\mathbb{A}_{[2]} := \frac{1}{2}\, dx^i \wedge dx^j \otimes \omega_{ij}$ with $\omega_{ij} \in \mathrm{End}_{C(M)}\mathcal{E}$ for all $i,\ j$ yields

$$P(\bar{\mathbb{A}}) = -2g^{ij}\, \mathbf{c}(dx^k \wedge dx^l)\, \omega_{ik}\omega_{jl} - g^{ij}g^{kl}\, \omega_{ik}\omega_{jl}. \tag{7}$$

According to (4) it is evident that the generalized Lichnerowicz formula (6) is the main tool to compute the non-commutative residue $res(D_{\mathbb{A}}^{-n+2})$ of a Dirac operator $D_{\mathbb{A}}$. Since it can be shown that $tr\big(\mathbf{c}(\mathbb{F}(\mathbb{A})^{\mathcal{E}/S})\big) = tr(\mathbb{A}_{[0]}^2)$ (cf. [A3]) these results imply

$$res(D_{\mathbb{A}}^{-n+2}) \;=\; -c_n \cdot \int_M *\, tr\big(\tfrac{1}{12} r_M\, \mathrm{I\!I}_{\mathcal{E}} + \mathbb{A}_{[0]}^2 + P(\bar{\mathbb{A}})\big). \tag{8}$$

[4]) A superconnection $\mathbb{A}\colon \Omega^*(M,E) \to \Omega^*(M,E)$ on a $\mathbb{Z}_2$-graded vector bundle $E = E^+ \oplus E^-$ can be defined as a sum $\mathbb{A} = \sum_{i\geq 0} \mathbb{A}_{[i]}$ of operators $\mathbb{A}_{[i]}\colon \Omega^*(M,E) \to \Omega^{*+i}(M,E)$ such that $\mathbb{A}_{[1]}$ is a connection on E which respects the grading and $\mathbb{A}_{[i]} \in \Omega^i(M, \mathrm{End}E)^-$ for $i \neq 1$. A superconnection $\mathbb{A}\colon \Omega^*(M,\mathcal{E})^{\pm} \to \Omega^*(M,\mathcal{E})^{\mp}$ on a Clifford module $\mathcal{E}$ is called a Clifford superconnection, if it is compatible with the Clifford action c, i.e. $[\mathbb{A}, c(a)] = c(\nabla a)$ for all $a \in \Gamma(C(M))$. For more details we recommend the recent book [BGV].

Notice that $res(D_{f^*\mathbb{A}}^{-n+2}) = res(D_{\mathbb{A}}^{-n+2})$ holds for all $f \in \mathrm{Aut}_{C(M)}\mathcal{E}$ which can be seen as the 'source' of gauge invariance. Computing (8) for our example $\mathbb{A} := \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \mathbb{A}_{[2]}$ mentioned above where $P(\bar{\mathbb{A}})$ is given by equation (7) we obtain

$$res(D_{\mathbb{A}}^{-n+2}) \;=\; -c_n \cdot \int_M \Big(\tfrac{rk(\mathcal{E})}{12} * r_M + tr\big(\mathbb{A}_{[0]} \wedge *\mathbb{A}_{[0]}\big) + 2tr\big(\mathbb{A}_{[2]} \wedge *\mathbb{A}_{[2]}\big)\Big). \tag{9}$$

As usual, $tr(\,\cdot \wedge * \cdot\,)$ denotes the bilinear form defined by the standard inner product $(\,\cdot \wedge * \cdot\,)$ on the exterior bundle $\Lambda^* T^* M$ combined with the complex trace tr on the endomorphism bundle $\mathrm{End}\,\mathcal{E}$. Obviously $tr(\,\cdot \wedge * \cdot\,)$ is symmetric but in general neither real nor definit.

Finally I turn back to particular geometric situation of theorem 1: Consider the odd endomorphism $J \in \mathrm{End}^-(\mathbb{C}^{1|1})$ which is defined by $J(z_1, z_2) = (z_2, -z_1)$ for all $(z_1, z_2) \in C^{1|1} \cong \mathbb{C} \oplus \mathbb{C}$. Then $J^2 = -\mathbb{1}_{\mathbb{C}^{1|1}}$ holds and obviously we have an extension $\mathcal{J} := \mathbb{1}_{\mathcal{E}} \hat{\otimes} J$ to the tensor bundle $\bar{\mathcal{E}} = \mathcal{E} \otimes \mathbb{C}^{1|1}$ with the same properties. There are natural 'even' and 'odd' inclusions $\iota, j \colon \mathrm{End}\mathcal{E} \hookrightarrow \mathrm{End}(\mathcal{E} \otimes \mathbb{C}^{1|1})$ defined by $\iota(F) := F \hat{\otimes} \mathbb{1}_{\mathbb{C}^{1|1}}$ and $j(F) := F \hat{\otimes} J$, respectively. Now assume that $\nabla^{\mathcal{E}}$ is a Clifford connection on $\mathcal{E}$ and let $\nabla^{\bar{\mathcal{E}}} := \left(\begin{smallmatrix} \nabla^{\mathcal{E}} & 0 \\ 0 & \nabla^{\mathcal{E}} \end{smallmatrix}\right)$ be the induced Clifford connection on $\bar{\mathcal{E}}$. To any odd endomorphism $\phi \in \Gamma(\mathrm{End}^-_{C(M)}\mathcal{E})$ we can define an even $\mathrm{End}_{C(M)}\bar{\mathcal{E}}$-vallued one form $\alpha := \sum_\mu dx^\mu \otimes j(\phi)$. Hence, $\widetilde{\nabla} := \nabla^{\bar{\mathcal{E}}} + \alpha$ is a Clifford connection and $\widetilde{\mathbb{A}} := \iota(\phi) + \widetilde{\nabla}$ is a Clifford superconnection on $\bar{\mathcal{E}}$ with twisting supercurvature $\mathbb{F}(\widetilde{\mathbb{A}})^{\bar{\mathcal{E}}/S} = \big(\widetilde{\nabla}^2\big)^{\bar{\mathcal{E}}/S} + \widetilde{\nabla}\iota(\phi) + \iota(\phi^2)$. We define

$$\mathbb{A} \;:=\; \iota(i\phi) \;+\; \nabla^{\bar{\mathcal{E}}} \;+\; \mathcal{J}\mathbb{F}(\widetilde{\mathbb{A}})^{\bar{\mathcal{E}}/S}. \tag{10}$$

Since $\mathbb{A}$ is a Clifford superconnection on $\bar{\mathcal{E}}$, the corresponding Dirac operator $D_\phi := D_{\mathbb{A}}$ is shown to verify theorem 1 by using (9), the decomposition $\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \mathbb{A}_{[2]}$ with

$$\mathbb{A}_{[0]} = \big(\iota(i\phi) + \mathcal{J}\iota(\phi^2)\big), \quad \mathbb{A}_{[1]} = \big(\nabla^{\bar{\mathcal{E}}} + \mathcal{J}\widetilde{\nabla}\iota(\phi)\big), \quad \mathbb{A}_{[2]} = \big(\mathcal{J}\iota(R^{\mathcal{E}/S}) + \mathcal{J}d^{\nabla^{\bar{\mathcal{E}}}}\alpha\big), \tag{11}$$

and the properties of $\mathcal{J}$. For more details I refer once more to [A3].

Let me now illustrate the power of theorem 1 when applied to particle physics by means of a 'toy model' which is strongly related to the celebrated Weinberg-Salam model of electroweak interaction:

We choose a four-dimensional closed Riemannian manifold M, for convinience endowed with a spin structure S and let $P_{G_{ew}} := M \times G_{\mathrm{ew}}$ be a trivial principal bundle with structure group $G_{\mathrm{ew}} := SU(2) \times U(1)$. To describe fermionic fields $\psi \in \Gamma(\mathcal{E})$ where $\mathcal{E} := S \otimes E$, let $E = P_{G_{\mathrm{ew}}} \times_\rho \mathbb{C}^2_L \oplus \mathbb{C}_R$ be the bundle associated to $P_{G_{\mathrm{ew}}}$ with respect to the representation $\rho(U,\, e^{i\theta})(v \oplus w) := U e^{iy_L\theta} v \oplus e^{iy_R\theta}$ for all $(U, e^{i\theta}) \in G_{\mathrm{ew}}$ and $(v \oplus w) \in \mathbb{C}^2_L \oplus \mathbb{C}_R$. Here $\mathbb{C}^2_L,\ \mathbb{C}_R$ correspond to left- and right-handed fermions with hypercharges $y_L,\ y_R \in \mathbb{Z}$, respectively [5]. Furthermore the Higgs-field $\varphi = \left(\begin{smallmatrix}\varphi_1 \\ \varphi_2\end{smallmatrix}\right) \in \Gamma(W)$ - with $W := M \times \mathbb{C}^2$

[5]) Suggestively we write $\psi_L = \left(\begin{smallmatrix}\nu_e \\ e_L\end{smallmatrix}\right) \in \Gamma(S \otimes \mathbb{C}^2_L)$, $\psi_R = e_R \in \Gamma(S \otimes \mathbb{C})$. To make things easier, notice that $N = 1$, i.e. we assume to have only one fermion generation.

being associated to $P_{G_{\rm ew}}$ with respect to the representation with hypercharge $y_\varphi \in \mathbb{Z}$ - is incorporated into the model by the linear map

$$\widetilde{\phi}\colon \mathrm{End}(M\times \mathbb{C}^2) \longrightarrow \mathrm{End}^-_{C(M)}\mathcal{E}, \qquad \varphi \mapsto \widetilde{\phi}(\varphi) := \begin{pmatrix} 0 & 0 & \varphi_1 \\ 0 & 0 & \varphi_2 \\ \bar{\varphi}_1 & \bar{\varphi}_2 & 0 \end{pmatrix} \tag{12}$$

which is determined by the Yukawa coupling $\phi(\varphi) = c_e\widetilde{\phi}(\varphi)$, i.e. the equality $(\psi, \phi(\varphi)\psi) = (\psi, c_e\widetilde{\phi}(\varphi)\psi)$ holds. Here the coupling constant $c_e \in \mathbb{R}$ gives rise to the mass of the electron m_e. Note that until now we have not fixed neither the hypercharges y_L, y_R for the fermions nor y_φ for the Higgs. Let $\nabla^W\colon \Gamma(W) \to \Gamma(T^*M \otimes W)$ be a connection and $\nabla^{\mathcal{E}}\colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ be a Clifford connection, both induced by a principal connection A on $P_{G_{\rm ew}}$. Interestingly one obtains (cf. [A4]):

Lemma 3. *Let $\nabla^{\mathcal{E}}\colon \Gamma(\mathrm{End}\mathcal{E}) \to \Gamma(T^*M \otimes \mathrm{End}\mathcal{E})$ also denote the connection on the endomorphism bundle $\mathrm{End}\mathcal{E}$. Then $tr\big(\nabla^{\mathcal{E}}\widetilde{\phi}(\varphi) \wedge *\nabla^{\mathcal{E}}\widetilde{\phi}(\varphi)\big) = 4(\nabla^W\varphi \wedge *\nabla^W\varphi)$ holds iff $y_\varphi = -y_L$ and $y_\varphi = y_L - y_R$.*

Let us now consider a variant of the Dirac operator $D = D_{\mathbb{A}}$ on $\bar{\mathcal{E}} = \mathcal{E} \otimes \mathbb{C}^{1|1}$ defined by the Clifford superconnection (10), more precisely the family $D_{(c_e,b_w,b_y)}$ of Dirac operators corresponding to the Clifford superconnections $\mathbb{A}_{(c_e,b_w,b_y)} := \iota(i\phi(\varphi)) + \nabla^{\bar{\mathcal{E}}} + \mathcal{J}\iota(B)\mathbb{F}(\widetilde{\mathbb{A}})^{\bar{\mathcal{E}}/S}$ uniquely determined by

$$\phi(\varphi) := c_e\widetilde{\phi}(\varphi) \in \mathrm{End}^-_{C(M)}\mathcal{E}, \quad B := \begin{pmatrix} b_w & 0 & 0 \\ 0 & b_w & 0 \\ 0 & 0 & b_y \end{pmatrix} \in \mathrm{End}^+_{C(M)}\mathcal{E}, \quad (c_e, b_w, b_y) \in \mathbb{R}^3. \tag{13}$$

Since it can be shown that $tr\big(\iota(i\phi(\varphi)) + \mathcal{J}\iota(B\phi(\varphi)^2)\big) = 2c_e^4 b_w b_y (\varphi,\varphi)^2 - 2c_e^2(\varphi,\varphi)$, one now uses lemma 3 and theorem 1 to obtain the

Corollary 4. *Let $\bar{\mathcal{E}} := \mathcal{E} \otimes \mathbb{C}^{1|1}$ be the particular Clifford module as defined above and $D_{(c_e,b_w,b_y)}\colon \Gamma(\bar{\mathcal{E}}) \to \Gamma(\bar{\mathcal{E}})$ be the family of Dirac operators corresponding to (13). Then*

$$res(D^{-2}_{(c_e,b_w,b_y)}) \sim \begin{cases} l_p^{-2}\int_M *r_M - A_w\int_M *tr(G_{\mu\nu}G^{\mu\nu}) \\ \;-A_y\int_M *tr(F_{\mu\nu}F^{\mu\nu}) + L\int_M *(\nabla^W_\mu\varphi, \nabla^W_\mu\varphi) \\ \;+K_1\int_M *(\varphi,\varphi)^2 - K_2\int_M *(\varphi,\varphi) \end{cases} \tag{14}$$

where G and F denote the $L(SU(2))$- and $L(U(1))$- components of the twisting curvature $R^{\mathcal{E}/S} \in \Omega^2(M, L(G_{\rm ew}))$ corresponding to the Clifford connection $\nabla^{\mathcal{E}}$ under the decomposition of the Lie algebra $L(G_{\rm ew}) = L(SU(2)) \oplus L(U(1))$ and with coefficients $A_w := \alpha_1 b_w^2$, $A_y := \alpha_1 b_y^2$, $L := 4\alpha_1 b_w b_y c_e^2$, $K_1 := \alpha_1 b_w b_y c_e^4$, $K_2 := \alpha_1 c_e^2$ with $\alpha_1 \sim a_1$.

Notice that we have introduced a physical lenght scale in units fixed by the Planck lenght $l_p := \sqrt{16\pi \mathrm{G}}$. Thus, (14) can be interpreted as the bosonic parts corresponding to (euclidean) Einstein-Yang-Mills- Higgs models (3) with one fermion generation which are classified by $(\alpha_1, b_w, b_y, c_e) \in \mathbb{R}^4$.

One immediate consequence of this corollary occurs if one 'switches off' the gravitational force, i.e. we are in flat space, and fixes the hypercharge of the left-handed fermions by $y_L = -1$. Hence lemma 3 implies $y_\varphi = 1$ and $y_R = -2$. So we are exactly in the case of the Weinberg-Salam model with one lepton generation (ν_e, e_L, e_R). With exception of the 'geometrical input', which, among others, concerns also the above mentioned combination of the hypercharges (y_L, y_R, y_φ), it is well-known that this model can be parametrized by

$$\big(g_w,\ g_y,\ c_e,\ \lambda,\ \mu\big) \in \mathbb{R}^5, \tag{15}$$

cf. [N]. Here $g_w,\ g_y$ are the coupling constants of the $SU(2)$ and $U(1)$ gauge fields, respectively, c_e is the Yukawa-coupling and $\lambda,\ \mu$ are the constants which enter into the Yang-Mills-Higgs lagrangian (2) via the Higgs potential $V(\varphi)$ as already mentioned. However, identification of our coefficients (A_w, A_y, L, K_1, K_2) to the standard ones

$$\widetilde{A}_w := \tfrac{1}{2g_w^2}, \quad \widetilde{A}_y := \tfrac{1}{4g_y^2} \quad \widetilde{L} := 1, \quad \widetilde{K}_1 := \tfrac{\lambda}{4}, \quad \widetilde{K}_2 := \tfrac{\mu^2}{2} \tag{16}$$

of the bosonic part of (2) [6]) yields

$$\lambda \ = \ \frac{1}{2\sqrt{2}}\, g_w \cdot g_y. \tag{17}$$

Thus, by our approach (14) we are indeed able to reduce the variables (15) of the model. In its turn λ enters into the relation $\frac{m_\varphi^2}{m_w^2} = \frac{8}{g_w^2}\cdot\lambda$ of the Higgs and the W-boson masses $m_\varphi,\ m_w$. Consequently we have $m_\varphi^2 = 2\sqrt{2}\cdot\frac{g_y}{g_w}\cdot m_w^2 = 2\sqrt{2}\cdot\tan\theta_w\cdot m_w^2$ where θ_w denotes the so-called Weinberg angle, cf. [N]. Since both $\sin^2\theta_w = 0.2325 \pm 0.0008$ and $m_w = 80.22\ (26)$ GeV are determined by scattering experiments, our simplified model provides us with the concrete (but unrealistic) prediction

$$m_\varphi \ \approx\ 100.1 \text{ GeV}. \tag{18}$$

So what have we learned until now ? Given a 'gauged' Clifford module $\mathcal{E}$ - i.e. a Clifford module which furnishes a representation $\rho\colon \mathcal{G} \to \mathrm{Aut}_{C(M)}\mathcal{E}$ of the group of gauge transformations $\mathcal{G}$ of a given principal bundle P_G with structure group G - and a Dirac-Yukawa operator $D_\varphi\colon \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, one should take $res(D_\phi^{-2})$ (where D_ϕ denotes the particular Dirac operator on $\bar{\mathcal{E}} := \mathcal{E}\otimes\mathbb{C}^{1|1}$ determined by (10) and (13), respectively) as a *definition* of the bosonic action S_b of an Einstein-Yang-Mills-Higgs model. Moreover it is easily verified that in terms of this Dirac operator D_ϕ the fermionic action S_f can be defined by $S_\mathrm{f} := \int_M *(\Psi, D_\phi\Psi)_{\bar{\mathcal{E}}}$ with $\Psi := \frac{1}{\sqrt{2}}\sum_i \psi\otimes e_i \in \Gamma(\bar{\mathcal{E}})$, $\{e_1,\ e_2\}$ denotes a orthonormal base of $\mathbb{C}^{1|1}$. Thus, we suggest to replace (3) by the *definition*

$$S := res(D_\phi^{-2}) \ +\ \int_M *(\Psi, D_\phi\Psi)_{\bar{\mathcal{E}}}, \tag{19}$$

[6]) Note that one has to take into account that $U(1)$ couplings are conventionally normalized differently than $SU(n)$ gauge couplings.

cf. [A3]. Writen in this most advanced form, the particular Dirac operator D_ϕ on $\bar{\mathcal{E}}$ can be identified as the origin of both gravitational and Yang-Mills gauge symmetries. At least in the case of our 'toy model' considered above, there is little extra information captured by this approach (19): On the geometrical side, calculation of the kinetic term for the Higgs involves constrains on the representations of the $U(1)$-part of the gauge group G_{ew} which result in the correct relations for the hypercharges y_L, y_R and y_φ. Moreover, the free variables of the model can be reduced which, in addition, constrains the Higgs mass m_φ to be 100.1 GeV, approximately. For the case of the full Standard model we refer to the forthcoming work [A4].

Conclusion

In this lecture I tried to show that Dirac operators on certain Clifford modules indeed could provide new insights into the origin of gauge symmetries in particle physics. Our approach (19) might also fit to 'unify', or at least connect, some different efforts in this field: Emphasizing the one-to-one correspondence of Dirac operators and Clifford superconnections there are relations to the Ne'eman-Sternberg approach to the Standard model [NS], for example the geometrical interpretation of the Higgs as a component of a superconnection. Furthermore, as our approach (19) to gravity coupled with matter relies completly on terms of differential geometry which have an analogue in the noncommutative world, (19) can be seen as a 'starting point' to define Einstein-Yang-Mills- Higgs models on noncommutative quantum spaces [C]. Note that the main idea concerning (19) can already be found in [AT1], [AT2]. Unfortunately in [AT2] the Dirac operator $\widetilde{D}$ is incorrect. However this will be treated elsewhere.

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